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A NOTE ON INJECTIVE $C(T)$-SPACES AND THE AMIR BOUNDARY

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ABSTRACT
We solve in the negative a problem of Wolfe if $C(T_A)$ is an injective Banach space whenever $C(T)$ is injective, T compact, and T_A is the Amir boundary of T (i.e., the complement of the maximal open extremally disconnected subset of T). In particular, we find T such that $C(T)$ is a P_3-space and $T_A \sim \beta\mathbf{N}\setminus\mathbf{N}$.

Wolfe [8, Th. 1.1] proved that if T is compact, $C(T)$ is an injective Banach space and T satisfies the countable chain condition (CCC), then $C(T_A)$ is injective as well. In fact, he proved that if $C(T)$ is a P_λ-space, then $C(T_A)$ is a $P_{\lambda-1}$-space. Then a little bit stronger result was obtained by the author [5, Th. 3.2] even for Fréchet injective spaces $C(T)$ with the compact-open topology, also under a CCC-like assumption on T, T locally compact.

Wolfe [8, Prob. 4.3] asked if for Banach $C(T)$-spaces the same result holds without the CCC assumption. We prove the following very negative answer.

THEOREM. *Let T be a compact space such that $C(T)$ is a P_λ-space. For every closed subset $B \subseteq T$, $T_A \subseteq B$, there is a compact space S such that S_A is homeomorphic to B and $C(S)$ is a $P_{\lambda+2}$-space.*

COROLLARY. *There is a compact set S such that $C(S)$ is injective while $C(S_A)$ is not injective (for example, $S_A \sim \beta\mathbf{N}\setminus\mathbf{N}$ and $C(S)$ is a P_3-space).*

PROOF OF COROLLARY. We apply the Theorem for $T = \beta\mathbf{N}$, $B = \beta\mathbf{N}\setminus\mathbf{N} \supseteq T_A = \emptyset$ (T is extremally disconnected). It is known that $C(\beta\mathbf{N}) \cong l_\infty$ is a P_1-space but $C(\beta\mathbf{N}\setminus\mathbf{N})$ is not injective [2, Cor. 2].

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REMARK. Amir [1, Th. 3] (see also [2, Th. 2], [3, Cor. 2]) showed that if T is compact and $C(T)$ injective, then T_A is nowhere dense in T. In [4, Ex. 4.2] the author gave an example of a completely regular topological space T with a fundamental sequence of compact sets such that $C(T)$ is injective Fréchet (in fact, $C(T) \simeq l_\infty^{\mathbf{N}}$) but $T_A = T$!

PROOF OF THEOREM. Let us consider the family of disjoint Čech–Stone compactifications of natural numbers $(\beta\mathbf{N}^{(x,i)})_{x\in B, i=0,1}$ and let $y^{(x,i)} \in \beta\mathbf{N}^{(x,i)}\setminus\mathbf{N}^{(x,i)}$ be fixed. Then let $\varphi: W \to T$ be the Gleason map (i.e., W is extremally disconnected and for any proper closed subset W_0 of W, $\varphi(W_0) \neq T$; existence of such a map is established in [7, Th. 3.2]) and let

$$X_0 := W \cup \left(\bigcup_{\substack{x\in B\\ i=0,1}} \beta\mathbf{N}^{(x,i)} \right), \qquad \text{with the disjoint sum topology;}$$

$$S_0 := T \cup \left(\bigcup_{\substack{x\in B\\ i=0,1}} \beta\mathbf{N}^{(x,i)}\setminus\{y^{(x,i)}\} \right).$$

The map $\Phi_0 : X_0 \to S_0$ is defined as follows:

$$\begin{aligned} \Phi_0(z) &:= \varphi(z) && \text{if } z \in W; \\ &= z && \text{if } z \in \beta\mathbf{N}^{(x,i)}\setminus\{y^{(x,i)}\}, \quad x \in B, \quad i = 0,1; \\ &= x && \text{if } z = y^{(x,i)}, \quad x \in B, \quad i = 0,1. \end{aligned}$$

We equip S_0 with the quotient topology (as easily seen S_0 is completely regular) and then we extend Φ_0 to a map $\Phi: X \to S$, where $X := \beta X_0$ and $S := \beta S_0$. It should be observed that $C(S)$ is isometrically embedded into $C(X)$ by the map $\hat{\Phi}: C(S) \to C(X)$, $\hat{\Phi}(f) := f \circ \Phi$. Similarly, using φ we can embed $C(T)$ into $C(W)$.

I. We will show that $C(S)$ is injective and belongs to $P_{\lambda+2}$-spaces, where $C(T)$ is a P_λ-space.

Let $P: C(W) \to C(T)$ be a projection, $\|P\| \leq \lambda + \epsilon$. Then we construct a projection $R: C(X) \to C(S)$, $\|R\| \leq 2 + \lambda + \epsilon$, as follows:

$$\begin{aligned} R(f)(z) &:= P(f|_W)(z) && \text{if } z \in T; \\ &= f(z) - f(y^{(x,i)}) + P(f|_W)(x) && \text{if } z \in \beta\mathbf{N}^{(x,i)}\setminus\{y^{(x,i)}\}; \end{aligned}$$

and then we extend in the unique way $R(f)$ on S. Now, it suffices to observe that X is extremally disconnected [6, 6.2.15] and thus $C(X)$ is a P_1-space.

II. We will show that $S_A = B$.

First, if $x \in B$, then

$$x \in (\overline{\mathbf{N}^{(x,0)}}^{S_0}) \cap (\overline{\mathbf{N}^{(x,1)}}^{S_0})$$

and thus $x \in S_A$ because $\mathbf{N}^{(x,i)}$ are open (discrete!) subsets both in S_0 and in S. Let $z \in S_0 \backslash B$, then there is a neighborhood of z in S_0 contained either in $T \backslash B$ if $z \in T \backslash B$ or in $\beta\mathbf{N}^{(x,i)} \backslash \{y^{(x,i)}\}$ if z belongs to this set. This implies that z does not belong to $\mathrm{bd}_{S_0} \bar{U}^{S_0}$ for any open subset U of S_0 and $(S_0)_A = B$.

Since B is compact, this completes the proof by the following lemma (cf. [6, Th. 6.2.5]):

Lemma. *If Z is a completely regular topological space, then* $(\beta Z)_A \subseteq \bar{Z}_A^{\beta Z}$.

Proof. If $z \in (\beta Z)_A \backslash \mathrm{cl}(Z_A)$, where cl means the closure in βZ, then there is an open subset U in βZ such that $\mathrm{cl}\, U \subseteq \beta Z \backslash \mathrm{cl}(Z_A)$ and $z \in \mathrm{bd}_{\beta Z}(\mathrm{cl}\, U)$. Thus,

$$V := Z \cap \mathrm{cl}(U \cap Z) = \mathrm{cl}_Z(U \cap Z)$$

is a closed-open set in Z for which closure in βZ coincides with $\mathrm{cl}\, U$. Indeed,

$$\mathrm{cl}\, U = \mathrm{cl}(U \cap Z) \subseteq \mathrm{cl}(Z \cap \mathrm{cl}\, U) = \mathrm{cl}(V) \subseteq \mathrm{cl}\, U.$$

Therefore, $\mathrm{cl}\, U$ is closed-open in βZ [6, 3.6.1, Cor. 4]; a contradiction.

Remark. If we apply the above construction to $T = \beta\mathbf{N}$ and $B = \beta\mathbf{N} \backslash \mathbf{N}$ (as in the Corollary), then we can prove using [9, Th. 1.4] that $C(S)$ is not a P_λ-space for any $\lambda < 3$ and we solve in the negative [9, Questions 2 and 11].

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